Solution to Homework Exercises No. 3.
FOURIER TRANSFORMS

(Notation: In several places we shall use the function $\chi_E : \mathbb{R} \to \mathbb{R}$, where E is some interval or other subset of E. This is defined by $\chi_E(s) = 1$ for all $s \in E$ and $\chi_E(s) = 0$ for all $s \notin E$, and is sometimes called the *characteristic function* of E or the *indicator function* of E.)

This is a rather long document, also because it contains a number of extra comments mainly intended for "experts" and "enthusiasts". *These are written like this in italics.*

1. **alef**. The graph of the function will be given in a separate file.

To check that $f \in G(\mathbb{R})$ we first have to check that it is piecewise continuous. The only points where f is not continuous are the integer points $x = 0, 1, 2, \ldots$ There are only finitely many of these points in any bounded interval, and at each of these points the two one sided limits both exist. The other condition which needs to be checked is that the limit $\lim_{R\to\infty} \int_{-R}^{R} |f(x)|dx$ is finite. For each $R > 1$, if m is some integer satisfying $m > R$ we have that

$$\begin{aligned}
\int_{-R}^{R} |f(x)|dx &= \int_{-R}^{0} |f(x)|dx + \int_{0}^{R} |f(x)|dx \\
&= 0 + \int_{0}^{R} |f(x)|dx \le \int_{0}^{m} |f(x)|dx = \sum_{n=0}^{m-1} \int_{n}^{n+1} |f(x)|dx \\
&= \sum_{n=0}^{m-1} \frac{1}{2^n} \le \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2.
\end{aligned}$$

So the limit as R tends to ∞ is finite and $f \in G(\mathbb{R})$.

bet. Since $f(x)$ is absolutely integrable, so is $e^{-i\omega x} f(x)$ for each constant $\omega \in \mathbb{R}$. So the limit $\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{-i\omega x} f(x)dx$ exists and defines the Fourier transform $F(\omega)$ of f at that value of ω. Since the limit exists it also equals the limit of the sequence $\left\{ \frac{1}{2\pi} \int_{-N}^{N} e^{-i\omega x} f(x)dx \right\}_{N\in\mathbb{N}}$. (But of course, in the other direction, the existence the limit of this sequence is NOT sufficient to ensure the existence of the Fourier transform.)

If $\omega = 0$ the same calculation as above shows that

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{2\pi} \int_{-N}^{N} e^{-i\omega x} f(x)dx &= \lim_{N\to\infty} \frac{1}{2\pi} \int_{-N}^{N} f(x)dx = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{n=0}^{N-1} \frac{1}{2^n} \\
&= \frac{1}{2\pi} \cdot 2 = \frac{1}{\pi}.
\end{aligned}$$

If $\omega \neq 0$ we have

$$\begin{aligned}
\int_{-N}^{N} e^{-i\omega x} f(x)dx &= \int_{0}^{N} e^{-i\omega x} f(x)dx = \sum_{n=0}^{N-1} \int_{n}^{n+1} e^{-i\omega x} f(x)dx \\
&= \sum_{n=0}^{N-1} \frac{1}{2^n} \int_{n}^{n+1} e^{-i\omega x} dx = \sum_{n=0}^{N-1} \frac{1}{2^n} \left(\frac{e^{-i\omega x}}{-i\omega} \Big|_{n}^{n+1} \right) \\
&= \sum_{n=0}^{N-1} \frac{i}{2^n \omega} \left(e^{-i\omega(n+1)} - e^{i\omega n} \right) = \sum_{n=0}^{N-1} \frac{i}{2^n \omega} e^{-i\omega n} \left(e^{-i\omega} - 1 \right) \\
&= \frac{i\left(e^{-i\omega} - 1\right)}{\omega} \sum_{n=0}^{N-1} \frac{1}{(2e^{i\omega})^n}
\end{aligned}$$

So

$$\begin{aligned}
F(\omega) &= \frac{i\left(e^{-i\omega} - 1\right)}{2\pi\omega} \sum_{n=0}^{\infty} \frac{1}{(2e^{i\omega})^n} = \frac{i\left(e^{-i\omega} - 1\right)}{2\pi\omega} \cdot \frac{1}{1 - \frac{1}{2e^{i\omega}}} \\
&= \frac{i\left(e^{-i\omega} - 1\right)}{2\pi\omega} \cdot \frac{2e^{i\omega}}{2e^{i\omega} - 1} = \frac{i\left(1 - e^{i\omega}\right)}{\pi\omega(2e^{i\omega} - 1)}.
\end{aligned}$$

gimel. To check if the function $F(\omega)$ is absolutely integrable, we first have to check its behaviour near $\omega = 0$. By L'Hôpital's rule it is easy to see that $\lim_{\omega\to 0} F(\omega) = F(0) = \frac{1}{\pi}$ so there is no problem near 0. (In fact, since $f \in G(\mathbb{R})$ there is a general theorem which ensures that $F(\omega)$ is continuous for all $\omega \in \mathbb{R}$.) But then we also have to check the behaviour of F for large ω. Intuitively it seems that $F(\omega)$ behaves like $\frac{1}{\omega}$ which is not absolutely integrable. To show that our intuition is correct needs a little bit more work because the factor $\left(e^{-i\omega} - 1\right)$ in the definition of F becomes very small for ω near $2\pi n$ for each integer n and we have to be sure that this "improvement" does not make the integral $\int_{-\infty}^{\infty} |F(\omega)|d\omega$ converge.

Here is one way to check exactly: For each positive integer N we have

$$\begin{aligned}
\int_{-N}^{N} |F(\omega)|d\omega &\geq \int_{2\pi}^{2\pi N} |F(\omega)|d\omega = \int_{2\pi}^{2\pi N} \left| \frac{\left(1 - e^{i\omega}\right)}{\pi\omega(2e^{i\omega} - 1)} \right| d\omega \\
&= \sum_{n=1}^{N-1} \int_{2\pi n}^{2\pi(n+1)} \frac{1}{|\omega|} \left| \frac{\left(1 - e^{i\omega}\right)}{\pi(2e^{i\omega} - 1)} \right| d\omega \\
&\geq \sum_{n=1}^{N-1} \int_{2\pi n}^{2\pi(n+1)} \frac{1}{2\pi(n+1)} \left| \frac{\left(1 - e^{i\omega}\right)}{\pi(2e^{i\omega} - 1)} \right| d\omega \\
&= \sum_{n=1}^{N-1} \frac{1}{2\pi(n+1)} \int_{2\pi n}^{2\pi(n+1)} \left| \frac{\left(1 - e^{i\omega}\right)}{\pi(2e^{i\omega} - 1)} \right| d\omega.
\end{aligned}$$

Since $e^{i\omega}$ is a 2π periodic function, we see that the integral $\int_{2\pi n}^{2\pi(n+1)} \left| \frac{\left(1-e^{i\omega}\right)}{\pi(2e^{i\omega}-1)} \right| d\omega$ has the same value $C = \int_{0}^{2\pi} \left| \frac{\left(1-e^{i\omega}\right)}{\pi(2e^{i\omega}-1)} \right| d\omega$ for all $n \in \mathbb{Z}$. It is also clear that $C > 0$.

So we see that

$$\int_{-N}^{N} |F(\omega)| d\omega \geq C \sum_{n=1}^{N-1} \frac{1}{2\pi(n+1)}. \tag{0.1}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, it follows that both expressions in (0.1) tend to $+\infty$ as N tends to $+\infty$. So F is not absolutely integrable on $\mathbb{R}$.

dalet. Despite the conclusion of part **gimel** we can still apply the theorem about the inverse Fourier transform. The function f has a left and right derivative at every point $x \in \mathbb{R}$. (Both derivatives are always zero.) So $\lim_{M\to\infty} \int_{-M}^{M} F(\omega) e^{i\omega x} d\omega = \frac{1}{2}(f(x+) + f(x-))$ for all $x \in \mathbb{R}$. This means that

$$g(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} = \frac{1}{2}(0+1) & \text{for } x = 0 \\ \frac{1}{2^n} & \text{for all } x \in (n, n+1), \text{ for each } n \in \mathbb{N} \\ \frac{3}{2^{n+1}} = \frac{1}{2}\left(\frac{1}{2^{n-1}} + \frac{1}{2^n}\right) & \text{for all } x = n, \text{ for each } n \in \mathbb{N}. \end{cases}$$

hey. By almost the same arguments as above we can see that f^2 is absolutely integrable and so

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^2 dx &= \lim_{N\to\infty} \int_{-N}^{N} |f(x)|^2 dx = \lim_{N\to\infty} \sum_{n=0}^{N-1} \left(\frac{1}{2^n}\right)^2 = \lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{1}{4^n} \\ &= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}. \end{aligned}$$

So we can apply Plancherel's theorem to obtain that

$$\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{2}{3\pi}.$$

vav. The function $g(x) = e^{-|x|}$ has Fourier transform $G(\omega) = \frac{1}{\pi(1+\omega^2)}$. So, by the generalized Plancherel formula,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{F(\omega)}{1+\omega^2} d\omega &= \pi \cdot \int_{-\infty}^{\infty} F(\omega) \overline{G(\omega)} d\omega = \pi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} f(x) e^{-|x|} dx. \end{aligned}$$

Since $|f(x)e^{-|x|}| \leq f(x)$ for all $x \in \mathbb{R}$ and f is absolutely integrable on $\mathbb{R}$, we get that the function $f(x)e^{-|x|}$ is also absolutely integrable on $\mathbb{R}$ and

$$= \frac{1}{2} \int_{-\infty}^{\infty} f(x) e^{-|x|} dx = \frac{1}{2} \int_{0}^{\infty} f(x) e^{-x} dx = \frac{1}{2} \lim_{N\to\infty} \int_{0}^{N} f(x) e^{-x} dx.$$

We can now calculate $\int_0^N f(x) e^{-x} dx$ in almost exactly the same way as we used above when calculating $\int_0^N f(x) e^{-i\omega x} dx$. We can even simply substitute $\omega = -i$ and use the result that we already obtained. This is because the formula $\int_a^b e^{cx} dx = \frac{e^{cx}}{c}\Big|_a^b = \frac{e^{cb} - e^{ca}}{c}$ is known to be true for all $c \neq 0$, real, imaginary and complex. (WHY?). So we simply substitute $\omega = -i$ in the previous obtained formula

$$\int_0^N e^{-i\omega x} f(x)dx = \frac{i\left(e^{-i\omega}-1\right)}{\omega} \sum_{n=0}^{N-1} \frac{1}{(2e^{i\omega})^n}$$

and get

$$\int_0^N e^{-x} f(x)dx = \frac{i\left(e^{-1}-1\right)}{-i} \sum_{n=0}^{N-1} \frac{1}{(2e)^n} = \left(1-e^{-1}\right) \sum_{n=0}^{N-1} \frac{1}{(2e)^n}.$$

Consequently

$$\int_{-\infty}^{\infty} \frac{F(\omega)}{1+\omega^2} d\omega = \frac{1}{2} \int_0^{\infty} e^{-x} f(x)dx = \frac{1}{2} \cdot \left(1-e^{-1}\right) \cdot \frac{1}{1-\frac{1}{2e}} = \frac{e-1}{2e-1}.$$

2. We want to apply the inverse Fourier transform to the given equation

$$F(\omega) + \int_{-\infty}^{\infty} F(\omega - x)e^{-x^2} dx = \int_{-\infty}^{\infty} \frac{x \cdot e^{-|\omega - x|}}{(1+x^2)^2} dx. \tag{0.2}$$

Let us first remark that the two integrals which appear in (0.2) are both convolutions. We know that the Fourier transform of $f * g$ satisfies

$$\mathcal{F}[f * g](\omega) = 2\pi F(\omega)G(\omega) \text{ for all } \omega \in \mathbb{R}, \tag{0.3}$$

where F and G are the Fourier transforms of f and g respectively and we assume that $f*g \in G(\mathbb{R})$. We need to use a similar (but NOT identical) result where inverse Fourier transforms replace Fourier transforms. We can prove this by copying the proof of (0.3) with a few small changes, or we can deduce it from (0.3) as follows:

Suppose that f, g, F, G and the convolution $F * G$ are all in $G(\mathbb{R})$ where F and G are the Fourier transforms of f and g respectively. Suppose also that f and g are continuous, and have left and right derivatives at every point of $\mathbb{R}$. Then $f(x) = \int_{-\infty}^{\infty} e^{ix\omega} F(\omega)d\omega$ and $g(x) = \int_{-\infty}^{\infty} e^{ix\omega} G(\omega)d\omega$ for all $x \in \mathbb{R}$. If u is the Fourier transform of F and v is the Fourier transform of G then (0.3) gives us that the Fourier transform of $F * G$ is $2\pi uv$, i.e.

$$2\pi u(x)v(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt}(F * G)(t)dt.$$

What we want is a formula for the INVERSE Fourier transform of $F * G$. We get this by replacing x by $-x$ in the above formula and multiplying by 2π. This gives

$$\int_{-\infty}^{\infty} e^{ixt}(F * G)(t)dt = 4\pi^2 u(-x)v(-x). \tag{0.4}$$

But there is also a simple connection between u and f. We have $u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} F(t)dt$. So $2\pi u(-x) = \int_{-\infty}^{\infty} e^{ixt} F(t)dt = f(x)$. Similarly, $2\pi v(-x) = g(x)$. So (0.4) implies that

$$\int_{-\infty}^{\infty} e^{ixt}(F * G)(t)dt = f(x)g(x). \tag{0.5}$$

Let us make some calculations using some known examples of Fourier transforms.

(i) If $G(x) = e^{-x^2}$ then its Fourier transform is $v(t) = \frac{1}{2\sqrt{\pi}} e^{-\frac{-t^2}{4}}$. So its inverse Fourier transform is

$$g(t) = 2\pi v(-t) = 2\pi \cdot \frac{1}{2\sqrt{\pi}} e^{-\frac{(-t)^2}{4}} = \sqrt{\pi} e^{-\frac{t^2}{4}}.$$

(ii) If $G(x) = e^{-|x|}$ then its Fourier transform is $v(t) = \frac{1}{\pi(1+t^2)}$. So its inverse Fourier transform is

$$g(t) = 2\pi v(-t) = \frac{2}{1+(-t)^2} = \frac{2}{1+t^2}.$$

(iii) Using integration by parts, and the fact that the Fourier transform of $e^{-|x|}$ is $\frac{1}{\pi(1+\omega^2)}$ we have that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\omega x} \frac{d}{dx}\left(\frac{1}{1+x^2}\right) dx &= 0 - \int_{-\infty}^{\infty} i\omega e^{i\omega x} \frac{1}{1+x^2} dx \\ &= \int_{-\infty}^{\infty} i\omega e^{i\omega x} \frac{1}{1+x^2} dx \\ &= i\omega \int_{-\infty}^{\infty} e^{i\omega x} \frac{1}{1+x^2} dx = i\omega\pi e^{-|\omega|}. \end{aligned}$$

So

$$\int_{-\infty}^{\infty} e^{i\omega x} \frac{-2x}{(1+x^2)^2} dx = i\omega\pi e^{-|\omega|}$$

and so

$$\int_{-\infty}^{\infty} e^{i\omega x} \frac{x}{(1+x^2)^2} dx = -\frac{1}{2} i\omega\pi e^{-|\omega|}.$$

Now we are ready to apply these calculations. Multiply both sides of (0.2) by $e^{i\omega x}$ and integrate with respect to ω on $(-\infty, \infty)$. Then formula (0.5) and the examples (i), (ii) and (iii) imply that

$$f(x) + f(x) \cdot \sqrt{\pi} e^{-\frac{x^2}{4}} = -\frac{1}{2} ix\pi e^{-|x|} \cdot \frac{2}{1+x^2}$$

and so

$$f(x) = \frac{-ix\pi e^{-|x|}}{(1+x^2)\left(1+\sqrt{\pi}e^{-\frac{x^2}{4}}\right)}.$$

Remark: Here is a slightly different way of solving the equation which you might find simpler. Instead of applying the inverse Fourier transform to both sides of (0.2), apply the Fourier transform itself. Then you can use the formula (0.3) directly, instead of having to find a modified version of it. You will have to use the Fourier transforms of the functions e^{-x^2} *and* $e^{-|x|}$ *and* $\frac{x}{(1+x^2)^2}$ *instead of their inverse Fourier transforms. Then, instead of getting a formula which contains* f *you will get a formula which contains the function* $g(t) = \frac{1}{2\pi}\int_{\infty}^{-\infty} e^{-it\omega} F(\omega) d\omega$. *But* $2\pi g(-t) = f(t)$ *so you can immediately also get a formula for* f *itself.*

3. Since the function f is odd, and vanishes outside the interval $[-a,a]$, its Fourier transform is given, for all $\omega \neq 0$, by

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-a}^{a} f(x)(\cos\omega x - i\sin\omega x)dx = -\frac{i}{2\pi}\int_{-a}^{a} f(x)\sin\omega x dx \\ &= -\frac{i}{2\pi}\cdot 2\int_0^a \sin\omega x dx = -\frac{i}{\pi}\left(-\frac{\cos\omega x}{\omega}\Big|_0^a\right) = \frac{i}{\pi\omega}(\cos\omega a - 1)\,. \end{aligned}$$

We also have $F(0)=0$. In a moment we will use the fact that this function F is also an odd function.

The function $f(x)$ has a left and a right derivative at every point $x\in\mathbb{R}$, so

$$\begin{aligned} \frac{1}{2}(f(x+)+f(x-)) &= \lim_{M\to\infty}\int_{-M}^{M} e^{ix\omega}\frac{i}{\pi\omega}(\cos\omega a - 1)\,d\omega \\ &= \lim_{M\to\infty}\int_{-M}^{M} (\cos\omega x + i\sin\omega x)\frac{i}{\pi\omega}(\cos\omega a - 1)\,d\omega \\ &= \lim_{M\to\infty}\int_{-M}^{M} i\sin\omega x\frac{i}{\pi\omega}(\cos\omega a - 1)\,d\omega \\ &= -\lim_{M\to\infty}\int_{-M}^{M} \sin\omega x\frac{\cos\omega a - 1}{\pi\omega}d\omega \\ &= -2\lim_{M\to\infty}\int_{0}^{M} \sin\omega x\frac{\cos\omega a - 1}{\pi\omega}d\omega. \end{aligned}$$

Now substitute $x=b$ in this equation. Then, multiplying by $-\pi/2$ and passing to the limit, we have

$$\int_0^\infty \sin b\omega\frac{\cos\omega a - 1}{\omega}d\omega = -\frac{\pi}{4}(f(b+)+f(b-))\,.$$

So the integral, call it $I(a,b)$, which we were asked to calculate is given by

$$I(a,b) = \begin{cases} 0 & \text{for } |b|>a, \text{ and also for } b=0 \\ -\frac{\pi}{4} & \text{for } b=a \\ \frac{\pi}{4} & \text{for } b=-a \\ -\frac{\pi}{2} & \text{for } b\in(0,a) \\ \frac{\pi}{2} & \text{for } b\in(-a,0). \end{cases}$$

4. aleph (i). These are standard results so I will not reproduce them here. (ii) The graph of ϕ will be given in a separate file.

bet. We know that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\omega}\chi_{[-b,b]}(x)dx = \frac{\sin\omega b}{\omega\pi}$ for each $b>0$. In particular we can choose $b=\pi$. Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\omega}\chi_{[-\pi,\pi]}(x)dx = \frac{\sin\omega\pi}{\omega\pi} = \phi(\omega\pi) \tag{0.6}$$

for all $\omega\in\mathbb{R}$. (The case when $\omega=0$ is easy to check separately.) Now replace ω by $\omega-n$ in (0.6). Then we get

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix(\omega-n)}\chi_{[-\pi,\pi]}(x)dx &= \frac{\sin(\omega-n)\pi}{(\omega-n)\pi} = \phi((\omega-n)\pi) \\ &= \phi((n-\omega)\pi) = \psi_n(\omega). \end{aligned} \tag{0.7}$$

The integral in this formula can be written as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix(\omega-n)}\chi_{[-\pi,\pi]}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\omega}e^{ixn}\chi_{[-\pi,\pi]}(x)dx.$$

So we see that ψ_n is the Fourier transform of the function $u_n(x) = e^{inx}\chi_{[-\pi,\pi]}(x)$. By the generalized Plancherel formula,

$$\begin{aligned} \langle \psi_m, \psi_n \rangle &= \int_{-\infty}^{\infty} \psi_m(\omega)\overline{\psi_n(\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_m(x)\overline{u_n(x)}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \end{aligned}$$

for all m, $n \in \mathbb{Z}$. This shows that $\{\psi_n\}_{n\in\mathbb{Z}}$ is an orthonormal system with respect to the given inner product.

gimel. Something suspicious and strange is going on here. But I am going to pretend not to notice it and work like a "robot". Later I will come back to show you where I did questionable things.

Since the Fourier transform of the function f that we have to work with is $\widehat{f}(\omega) = \omega^2\chi_{[-\pi,\pi]}$, the function f itself must be given by

$$f(x) = \lim_{M\to\infty}\int_{-M}^{M} e^{i\omega x}\widehat{f}(\omega)d\omega = \int_{-\pi}^{\pi} e^{i\omega x}\omega^2 d\omega. \tag{0.8}$$

We will calculate this integral later. In any case it is clear from this formula that $f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}2\pi(-t)^2dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}2\pi t^2\chi_{[-\pi,\pi]}(t)dt$, i.e. f is the Fourier transform of the function $v(t) = 2\pi t^2\chi_{[-\pi,\pi]}(t)$.

Let us calculate the generalized Fourier coefficients of the given function f with respect to the orthonormal system $\{\psi_n\}_{n\in\mathbb{Z}}$. Using the generalized Plancherel formula, we have, for each $n \in \mathbb{Z}$,

$$c_n = \langle f, \psi_n \rangle = \frac{1}{2\pi}\langle v, u_n \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\omega^2\chi_{[-\pi,\pi]}(\omega)\overline{e^{in\omega}\chi_{[-\pi,\pi]}(\omega)}d\omega = \int_{-\pi}^{\pi}\omega^2 e^{-in\omega}d\omega.$$

If $n = 0$ this gives

$$c_0 = \int_{-\pi}^{\pi}\omega^2 d\omega = \left.\frac{\omega^3}{3}\right|_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

and for all $n \neq 0$,since ω^2 is an even function, we have

(0.9)
$$c_n = \int_{-\pi}^{\pi}\omega^2 e^{-in\omega}d\omega = \int_{-\pi}^{\pi}\omega^2\cos n\omega d\omega - i\int_{-\pi}^{\pi}\omega^2\sin n\omega d\omega = \int_{-\pi}^{\pi}\omega^2\cos n\omega d\omega + 0.$$

After performing integration by parts twice we get that

$$\int_{-\pi}^{\pi}\omega^2\cos n\omega d\omega = 2\frac{n^2(\sin n\pi)\pi^2 + 2n(\cos n\pi)\pi - 2\sin n\pi}{n^3}. \tag{0.10}$$

Note, for our convenience later, that these calculations hold for every real $n \neq 0$, even if it is not an integer. This means that, substituting x in place of n, we have that the integral in (0.8) satisfies

(0.11)

$$\int_{-\pi}^{\pi} e^{i\omega x}\omega^2 d\omega. = \int_{-\pi}^{\pi} \omega^2 \cos x\omega d\omega = 2\frac{x^2(\sin x\pi)\,\pi^2 + 2x(\cos x\pi)\,\pi - 2\sin x\pi}{x^3}.$$

Since, for now, we are assuming that n IS an integer, we obtain from (0.10) that

$$c_n = 2\frac{0 + 2n(-1)^n\,\pi - 0}{n^3} = \frac{4\pi(-1)^n}{n^2}.$$

So the generalized Fourier series of f with respect to the functions $\psi_n(x) = \frac{\sin(\pi(x-n))}{\pi(x-n)}$ is

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} c_n\psi_n(x) \\ &= \frac{2\pi^3}{3}\cdot\frac{\sin \pi x}{\pi x} + \sum_{n=1}^{\infty}\frac{4\pi(-1)^n}{n^2}\left(\frac{\sin(\pi(x-n))}{\pi(x-n)} + \frac{\sin(\pi(x+n))}{\pi(x+n)}\right). \qquad (0.12) \end{aligned}$$

I used the symbol $\sim$ because, analogously to what happens with usual Fourier series, we do not know at first if the series converges, and even if it does converge, we do not know if it actually converges to $f(x)$.

If you got to here, then you have solved part ***gimel*** *of the problem sufficiently well for the requirements of this course. But at the end of these pages we will give some additional comments for students who may want to understand a bit more deeply. This will include a proof that the series (0.12) does in fact converge and its sum is $f(x)$.*

dalet. (i) We have already done this. From (0.8) and (0.9) we can see that $f(n) = c_n$ for each $n \in \mathbb{Z}$, i.e. $f(0) = \frac{2\pi^3}{3}$, and, for $n \neq 0$, $f(n) = \frac{4\pi(-1)^n}{n^2}$.

(ii) It is not easy to draw the graph of f. It is continuous, and tends to 0 at $\pm\infty$ and does quite a lot of oscillating. We will show its graph in a separate file, but with the help of a computer program.

hey. (i) We were told that the fourier transform of f is $\omega^2\chi_{[-\pi,\pi]}$, i.e. $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}f(x)dx = \omega^2\chi_{[-\pi,\pi]}(\omega)$. In particular, if we set $\omega = 0$, this gives us that $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)dx = 0$.

But is this really correct? So far we have only checked that the integral $\int_{-\infty}^{\infty} f(x)dx$ only exists in the "P.V." sense, i.e. we can calculate $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$ and indeed this limit does equal 0. (See (0.18) in the remarks at the end of these pages.) We have not yet checked to see if the limits $\lim_{R\to\infty}\int_0^R f(x)dx$ and $\lim_{R\to\infty}\int_{-R}^0 f(x)dx$ exist. But we will do that now. We are lucky that our function f is an even function, and so $\int_0^R f(x)dx = \int_{-R}^0 f(x)dx = \frac{1}{2}\int_{-R}^R f(x)dx$. So these two limits DO exist. This is enough to ensure that the limit

$$\lim_{A\to-\infty, B\to+\infty}\int_A^B f(x)dx$$

exists, i.e. $\int_{-\infty}^{\infty} f(x)dx$ is defined in a stronger way than just "P.V."

(ii) Since f is the Fourier transform of $v(t) = 2\pi t^2 \chi_{[-\pi,\pi]}(t)$ we can apply Plancherel's formula:

$$\begin{aligned}\int_{-\infty}^{\infty} |f(\omega)|^2 d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |v(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\pi^2 x^4 dx \\ &= 2\pi\left(\left.\frac{x^5}{5}\right|_{-\pi}^{\pi}\right) = \frac{4\pi^6}{5}.\end{aligned}$$

We could alternatively use the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \left|\widehat{f}(\omega)\right|^2 d\omega$$

which happens to be true, even though f is not in $G(\mathbb{R})$. (See Remark 1 below.)

5. **alef**. By the Cauchy-Schwartz inequality

$$\begin{aligned}\left|\int_{-\infty}^{\infty} tf(t)f'(t)dt\right| &\le \sqrt{\int_{-\infty}^{\infty} |tf(t)|^2 dt \cdot \int_{-\infty}^{\infty} |f'(t)|^2 dt} \\ &= \Delta_t \sqrt{\int_{-\infty}^{\infty} |f'(t)|^2 dt}.\end{aligned} \tag{0.13}$$

We have $f' \in G(\mathbb{R})$ and the Fourier transform of f' is given by

$$\begin{aligned}\widehat{(f')}(\omega) &= \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R} e^{-i\omega t} f'(t)dt \\ &= \frac{1}{2\pi}\lim_{R\to\infty}\left(e^{-i\omega t} f(t)\Big|_{-R}^{R} - \int_{-R}^{R} -i\omega e^{-i\omega t} f(t)dt\right) \\ &= \frac{1}{2\pi}\left(0 + \int_{-\infty}^{\infty} i\omega e^{-i\omega t} f(t)dt\right) = i\omega F(\omega).\end{aligned}$$

So Plancherel's formula gives that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f'(t)|^2 dt = \int_{-\infty}^{\infty} |\omega F(\omega)|^2\,d\omega = \Delta_\omega^2.$$

We substitute this in (0.13) to get

$$\left|\int_{-\infty}^{\infty} tf(t)f'(t)dt\right|^2 \le \Delta_t^2 2\pi \Delta_\omega^2 \tag{0.14}$$

as required.

bet.

$$\begin{aligned}\int_A^B tf(t)f'(t)dt &= \frac{1}{2}\int_A^B t\frac{d}{dt}(f(t))^2\,dt \\ &= \frac{1}{2}\left(t\,(f(t))^2\Big|_A^B - \int_A^B (f(t))^2 dt\right).\end{aligned}$$

When we take the limit as B tends to $+\infty$ and A tends to $-\infty$ this gives

$$\int_{-\infty}^{\infty} tf(t)f'(t)dt = \frac{1}{2}\left(0 - \int_{-\infty}^{\infty} (f(t))^2 dt\right) = \frac{1}{2}(-2\pi) = -\pi. \tag{0.15}$$

Hey, wait a minute! What if f has complex values? This formula is true only when f takes real values. If f is complex we can prove instead that $\int_{-\infty}^{\infty} tf(t)\overline{f'(t)}dt = -\pi$, using some small modifications of the previous argument. Also in the formulation of this question we forgot to give the condition $\lim_{t\to-\infty}\sqrt{|t|}f(t) = 0$ which is needed for treating the limit as A tends to $-\infty$. Sorry!

gimel. This is just a simple combination of the results of aleph and **bet**. Substitute (0.15) in (0.14) to get $\pi^2 \le \Delta_t^2 2\pi\Delta_\omega^2$ and then divide by 2π.

This works for real valued f. But we can also deal with complex valued f since almost exactly the same argument as we used in aleph also shows that $\left|\int_{-\infty}^{\infty} tf(t)\overline{f'(t)}dt\right|^2 \le 2\pi\Delta_t^2\Delta_\omega^2$.

dalet. Suppose that there exists a function f with all the given properties.

Sorry, again, there is a misprint: The condition $F(\omega) \le 1$ should be $|F(\omega)| \le 1$.

Then $\Delta_t^2 = \int_{-\infty}^{\infty} |tf(t)|^2 dt = \int_{-1}^{1} |tf(t)|^2 dt \le \int_{-1}^{1} |t|^2 dt = 1$.

Similarly, $\Delta_\omega^2 = \int_{-\infty}^{\infty} |\omega F(\omega)|^2 d\omega = \int_{-1}^{1} |\omega F(\omega)|^2 d\omega \le \int_{-1}^{1} |\omega|^2 d\omega = 1$.

Then, using **gimel**, we get $\frac{\pi}{2} \le \Delta_t^2\Delta_\omega^2 \le 1$ which is of course impossible. So no such function f can exist.

(Results like this have a deep meaning in physics where they can be used to show ("uncertainty principle") that if we know the position of a particle with high precision then we only have very approximate information about the momentum of the particle, and vice versa.)

hey. (For simplicity, here again we will assume that f is a real valued function.) Following the hint, if we want the Cauchy-Schwartz inequality which we used in part aleph to hold with equality, then we have to take $f'(t) = \lambda t f(t)$ for all $t \in \mathbb{R}$ and some constant λ.

If we know that f does not vanish anywhere, then we can divide by f and integrate:

$$\frac{f'(t)}{f(t)} = \lambda t \Longrightarrow \ln|f(t)| = \frac{\lambda t^2}{2} + C \Longrightarrow |f(t)| = e^{\frac{\lambda t^2}{2}+C}.$$

If we do not want to make any assumptions in advance about f then instead we can use an "integrating factor" (as in one of the general methods for solving first order linear differential equations).

$$\begin{aligned} f'(t) - \lambda t f(t) &= 0 \Longrightarrow e^{-\frac{\lambda t^2}{2}}\left(f'(t) - \lambda t f(t)\right) = 0 \\ &\Longrightarrow \frac{d}{dt}\left[e^{-\frac{\lambda t^2}{2}} f(t)\right] = 0 \Longrightarrow e^{-\frac{\lambda t^2}{2}} f(t) = M \end{aligned}$$

where M is a constant. So $f(t) = Me^{\frac{\lambda t^2}{2}}$ for suitable constants M and λ. Now let us find all possible values for M and λ. We use the condition $\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(t)|^2 dt = 1$, which now gives that $\frac{M^2}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t^2} dt = 1$. Clearly λ must be a negative number. We set $\lambda = -\alpha^2$ and make the change of variables $x = \alpha t$. Then

$$\frac{2\pi}{M^2} = \int_{-\infty}^{\infty} e^{-(\alpha t)^2} dt = \frac{1}{\alpha}\int_{-\infty}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{\alpha}.$$

This shows that $M^2 = 2\alpha\sqrt{\pi}$. We conclude that any real function f which satisfies the conditions given at the beginning of Question 5 also satisfies the equality $\frac{\pi}{2} =$

$\Delta_t^2\Delta_\omega^2$ if and only if it is of the form

$$f(t) = Me^{-\frac{\alpha^2 t^2}{2}}$$

for some constant $\alpha > 0$ and some constant $M = \pm\sqrt{2\alpha\sqrt{\pi}}$. (It is maybe a bit more elegant to write $f(t) = \pm\sqrt[4]{8\beta\pi}\cdot e^{-\beta t^2}$ for some $\beta > 0$.)

SOME MORE DETAILED COMMENTS ABOUT PART GIMEL OF QUESTION 4.

We were told that the function f that we have to work with has Fourier transform $\widehat{f}(\omega) = \omega^2\chi_{[-\pi,\pi]}$. But this function $\widehat{f}$ is not continuous at the points $\omega = \pm\pi$. This means that f cannot be in $G(\mathbb{R})$. So how can we talk about its Fourier transform? First we should ask the people who wrote this question what they mean? (None of you did this yet!) And then we can try to use some intuition and imagination to see if it is possible to somehow define the Fourier transform for functions f which are not in $G(\mathbb{R})$.

We pretended above that we did not notice this problem and we wrote the formula (0.8) for the inverse Fourier transform of $\widehat{f}$. We have not proved that this formula is true when f is not in $G(\mathbb{R})$ but we have shown (see (0.11)) that it implies that

$$f(x) = 2\frac{x^2(\sin\pi x)\,\pi^2 + 2x(\cos\pi x)\,\pi - 2\sin\pi x}{x^3} \tag{0.16}$$

for all $x \neq 0$, and, for $x = 0$, that

$$f(0) = \int_{-\pi}^{\pi}\omega^2 d\omega = \frac{2}{3}\pi^3. \tag{0.17}$$

The function $f(x)$ obtained in this way is continuous for all x. (This can be checked at $x = 0$ by using L'Hôpital's rule, but it can also be seen almost immediately from the formula $f(x) = \int_{-\pi}^{\pi}\omega^2\cos\omega x\,d\omega$.) But f is not absolutely integrable on $\mathbb{R}$, since for large values of x it is approximately equal to $\frac{2\pi^2\sin\pi x}{x}$. Apparently the person who wrote this question intended that we should take f to be the function DEFINED by (0.8) and so given by (0.16) and (0.17), even though it is not in $G(\mathbb{R})$. (See Remark 1 below for further comments about this.)

Remark 1. (Generalizing the definition of the Fourier transform.) We had hoped that the function f defined by (0.8) and therefore given by (0.16) and (0.17) would be the function that we are looking for, whose fourier transform is $\omega^2\chi_{[-\pi,\pi]}(\omega)$. But since this function f is not in $G(\mathbb{R})$ we perhaps cannot define its Fourier transform in the usual way. On the other hand the given function $\widehat{f}(x) = x^2\chi_{[-\pi,\pi]}(x)$ IS in $G(\mathbb{R})$ and it has a right and left derivative at every point. Also, our definition of f means that the Fourier transform $\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\widehat{f}(x)dx$ of the function $\widehat{f}$ is $\frac{1}{2\pi}f(-t) = \frac{1}{2\pi}f(t)$. This means that, for all $x \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{2}\left(\widehat{f}(x+) + \widehat{f}(x-)\right) &= \lim_{R\to\infty}\int_R^{-R} e^{itx}\frac{1}{2\pi}f(-t)dt \\ &= \frac{1}{2\pi}\lim_{R\to\infty}\int_R^{-R} e^{-itx}f(t)dt. \end{aligned} \tag{0.18}$$

So, except at the two points $x = \pm\pi$, if we use a generalized version of the Fourier transform, where "$\int_{-\infty}^{\infty}$" is replaced by "P.V. $\int_{-\infty}^{\infty}$" which means exactly

"$\lim_{R\to\infty}\int_R^{-R}$" then we do get that $\widehat{f}$ is the "generalized" Fourier transform of f. Of course this generalized transform may not always have exactly the same properties that we showed to hold in the case where the function f is in $G(\mathbb{R})$. (Of course, for functions which ARE in $G(\mathbb{R})$, the new and old definitions give the same transform.)

More generally, let me mention that there are also certain special ways of defining Fourier transforms of functions which are much 'worse" than the particular function f defined by (0.8). There are even ways to define Fourier transforms of "things" (measures, distributions) which are not even functions.

One important case when this can be done is for all piecewise continuous functions $u:\mathbb{R}\to\mathbb{C}$ with the property that $|u|^2$ is absolutely integrable. Although some of these functions may not be in $G(\mathbb{R})$ (for example consider $u(x)=\frac{1}{1+|x|}$), it is possible to define their Fourier transforms by a suitable limiting process. These transforms may fail to be continuous and it can also happen that they do not tend to 0 at $\pm\infty$. But they still satisfy the Plancherel formula, and any two such functions satisfy the generalized Plancherel formula.

Remark 2. (The series (0.12).) To study this particular series (0.12), we observe that, since $\left|\frac{\sin x}{x}\right|\le 1$, we have $|\psi_n(x)|\le 1$ for all x and all n. We also know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. Combining these facts shows that the series converges absolutely. We can also apply the Weierstrass "m-test" to see that this convergence is uniform on $\mathbb{R}$.

The numbers c_n which we calculated above are the complex Fourier coefficients of the function $u:[-\pi,\pi]\to\mathbb{R}$ defined by $u(x)=2\pi x^2$. Because of the properties of this function, the Fourier series $\sum_{n=-\infty}^{\infty}c_ne^{inx}$ converges to $u(x)$ uniformly on $[-\pi,\pi]$. So, for every fixed $\omega\in\mathbb{R}$; we also have that the series $\sum_{n=-\infty}^{\infty}e^{-i\omega x}c_ne^{-inx}$ converges to $e^{-i\omega x}u(x)$ uniformly on $[-\pi,\pi]$. Consequently

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}e^{-i\omega x}\sum_{n=-N}^{N}c_ne^{inx}dx=\int_{-\pi}^{\pi}e^{-i\omega x}u(x)dx.$$

This can be rewritten as

$$\lim_{N\to\infty}\sum_{n=-N}^{N}c_n\int_{-\infty}^{\infty}\chi_{[-\pi,\pi]}(x)e^{-i\omega x}e^{inx}dx=\int_{-\infty}^{\infty}e^{-i\omega x}\chi_{[-\pi,\pi]}(x)u(x)dx.$$

Now let us divide both sides of this equation by 2π and substitute from (0.7) for the left side. Then we get

$$\sum_{n=-\infty}^{\infty}c_n\psi_n(\omega)=\int_{-\infty}^{\infty}e^{-i\omega x}\chi_{[-\pi,\pi]}(x)x^2dx.$$

The right side is the function f which we DEFINED using (0.8).